Features of (Un)decidable Logics

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Plan for the talk

- (Un)decidability: what and why?
- Propositional team logics and their decidability
- Exploring boundaries between the decidable and the undecidable
    - Solving problems and obtaining insights along the way
    - Using insights to solve one last problem

(Un)decidability: what and why?

What?

A decision problem is a collection of inputs I, with a yes-or-no question for each $i \in I$.

A decision problem is decidable if there is an effective method that, given any $i \in I$, accurately answers the question. Otherwise, it is undecidable.

A logic $\mathbf{L}$, in a language $\mathcal{L}$, is decidable if there is an effective method that, given any $\varphi \in \mathcal{L}$, determines whether $\varphi \in \mathbf{L}$. Otherwise, it is undecidable.

Why?

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Why? *Because it is a deep, profound and significant conceptual distinction.*

Propositional team logics and their decidability

Traditionally (in, e.g., CPC), formulas φ are evaluated at single valuations $v : \mathbf{Prop} \to \{0,1\}$,

$$v \vDash \varphi.$$

In team semantics, formulas φ are evaluated at sets ('teams') of valuations $s \subseteq \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$,

$$s \vDash \varphi.$$

Definition (some team-semantic clauses)

Let $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$. For $s \in \mathcal{P}(X)$, we define

$s \vDash p$	iff	$\forall v \in s : v(p) = 1$,
$s \vDash \varphi \wedge \psi$	iff	$s \vDash \varphi$ *and* $s \vDash \psi$,
$s \vDash \varphi \varovee \psi$	iff	$s \vDash \varphi$ *or* $s \vDash \psi$,
$s \vDash {\sim}\varphi$	iff	$s \nvDash \varphi$,
$s \vDash \varphi \vee \psi$	iff	there exist $s', s'' \in \mathcal{P}(X)$ such that $s' \vDash \varphi$; $s'' \vDash \psi$; and $s = s' \cup s''$.

Observation. All propositional team logics are decidable: given φ, simply check whether $s \vDash \varphi$ for all $s \subseteq \{v \mid v : \mathbf{Prop}(\varphi) \to \{0,1\}\}$.

Yet, this explanation is hardly satisfactory.
What is it that makes propositional team logics decidable, *and others not?*

Team semantics as relational semantics

Recall our semantic clauses: For $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$ and $s \in \mathcal{P}(X)$, we had

$$
\begin{array}{lll}
s \vDash p & \textbf{iff} & \forall v \in s : v(p) = 1, \\
s \vDash \varphi \wedge \psi & \textbf{iff} & s \vDash \varphi \ \textit{and} \ s \vDash \psi, \\
s \vDash \varphi \mathbin{\mathbb{V}} \psi & \textbf{iff} & s \vDash \varphi \ \textit{or} \ s \vDash \psi, \\
s \vDash {\sim}\varphi & \textbf{iff} & s \nvDash \varphi, \\
s \vDash \varphi \vee \psi & \textbf{iff} & \text{there exist } s', s'' \in \mathcal{P}(X) \text{ such that } s' \vDash \varphi; \\
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&s \vDash \neg\varphi && \text{iff} && s \nvDash \varphi,\\
&s \vDash \varphi \circ \psi && \text{iff} && \text{there exist } s', s'' \in \mathcal{P}(X) \text{ such that } s' \vDash \varphi;\\
& && && s'' \vDash \psi; \text{ and } s = s' \cup s''.
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This induces a powerset frame $\mathbb{F} = (\mathcal{P}(X), \cup)$, where '$\circ$' is a binary modality referring to the ternary $\cup$-relation $s = s' \cup s''$;

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This induces a powerset frame $\mathbb{F} = (\mathcal{P}(X), \cup)$, where '$\circ$' is a binary modality referring to the ternary $\cup$-relation $s = s' \cup s''$; and a model $\mathbb{M} = (\mathcal{P}(X), \cup, V)$ with a 'principal valuation', i.e.,

$$V(p) := \{s \in \mathcal{P}(X) \mid \forall v \in s : v(p) = 1\} = \downarrow\{v \in X \mid v(p) = 1\}.$$

In fact, if we take all powerset frames $(\mathcal{P}(X), \cup)$, redefine the base clause

$$(\mathcal{P}(X), \cup, V), s \Vdash p \qquad \text{iff} \qquad s \in V(p),$$

and only allow principal valuations $V : \mathbf{Prop} \to \{\downarrow s \mid s \in \mathcal{P}(X)\}$, we get sound and complete relational semantics for team logics.

Proof. A simple p-morphism argument.

Powerset frames and Boolean frames

Summarizing, (i) team logics are decidable, and (ii) relational semantics for team logics are given by powerset frames $(\mathcal{P}(X), \cup)$ with principal valuations $V : \mathbf{Prop} \to \{\downarrow s \mid s \in \mathcal{P}(X)\}$.

Question: *Sticking with the signature* $\{\wedge, \vee, \neg, \circ\}$, *what happens if we allow for arbitrary valuations* $V : \mathbf{Prop} \to \mathcal{P}\mathcal{P}(X)$? *Does the logic remain decidable?*

In fact, this question is intimately related with an open problem: Goranko and Vakarelov (1999) consider the logic of Boolean frames – instead of a powerset $\mathcal{P}(X)$, the carrier is a Boolean algebra B – and raises the problem of its decidability.[1]

Theorem

The logic of powerset frames, in the signature $\{\wedge, \vee, \neg, \circ\}$, with arbitrary valuations is **undecidable**. And so is the hyperboolean modal logic of Goranko and Vakarelov (1999).

[1] Goranko and Vakarelov (1999) call their logic 'hyperboolean modal logic' and include modalities for all the Boolean operations, not just the join.

Proof method: tiling

- A (Wang) tile is a square with colors on each side.
- The tiling problem: given any finite set of tiles $\mathcal{W}$, determine whether each point in the quadrant $\mathbb{N}^2$ can be assigned a tile from $\mathcal{W}$ such that neighboring tiles share matching colors on connecting sides.
- The tiling problem was introduced by Wang (1963) and proven undecidable by Berger (1966).

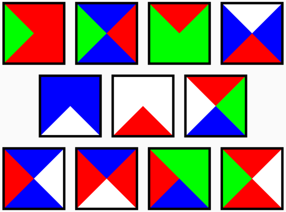

Figure 1: Wang tiles

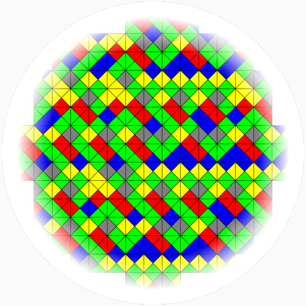

Figure 2: A tiling of the plane

Figures taken from: https://en.wikipedia.org/wiki/Wang_tile

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Proof idea.

For each finite set of tiles $\mathcal{W}$, we construct a formula $\phi_{\mathcal{W}}$ such that $\mathcal{W}$ tiles the quadrant if and only if $\phi_{\mathcal{W}}$ is satisfiable. □

Dividing the proof into two lemmas, corresponding to a direction each, we can prove both results in one go:

Lemma

If $\phi_{\mathcal{W}}$ is satisfiable (in a Boolean frame), then $\mathcal{W}$ tiles $\mathbb{N}^2$.

Lemma

If $\mathcal{W}$ tiles $\mathbb{N}^2$, then $\phi_{\mathcal{W}}$ is satisfiable (in $(\mathcal{P}(\mathbb{N}), \cup)$).

Insight 1: valuations matter

Semilattice frames, associativity and negation

Question: *Since we can weaken from powersets to Boolean algebras and stay undecidable, how much further can we go while remaining undecidable?*

Weakening from powersets $(\mathcal{P}(X), \cup)$ to general (join-)semilattices $(S, \sqcup)$, we get a problem posed by Bergman (2018) and Jipsen et al. (2021) (and SBK (2023a)).

Theorem

For any class $\mathcal{S}$ of semilattices containing $(\mathcal{P}(\mathbb{N}), \cup)$, its logic in the signature $\{\wedge, \vee, \neg, \circ\}$, is undecidable.

Proof. *See Blackboard*

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Question: *What if we weaken even further than semilattices?*

(Partial) answer: As semilattices are partial orders '$\leq$' with all binary suprema, we could consider the logic of all *partial orders simpliciter*. This is modal information logic, which is proven decidable in SBK (2023b).[2]

Question: *What if we, instead, reduce our signature $\{\wedge, \vee, \neg, \circ\}$?*

Answer: If we stick to semilattices but *omit negation*, so signature is $\{\wedge, \vee, \circ\}$, we obtain *Finean truthmaker semantics*, proven decidable in SBK (2023a).

[2] This will be part of my talk at Tsinghua University on Thursday.

Insight 2: associativity matters

Insight 3: negation matters

(Un)decidability of relevant S: using our insights

Problem of concern: *Is relevant logic* $\mathbf{S}$ *decidable?*

$\mathbf{S}$ is the logic of semilattice frames $(S, \sqcup, \mathbf{0})$ with a bottom element $\mathbf{0}$, with arbitrary valuations, in the signature $\{\wedge, \vee, \rightarrow\}$. '$\rightarrow$' is closely connected to '$\circ$' (it is its residual).

What we know about the problem:

- Omitting disjunction, the logic $\mathbf{S}_{\wedge,\rightarrow}$ is decidable.
- If we restrict to hereditary valuations, we obtain positive intuitionistic logic, which is decidable.
- $\mathbf{S}$ is closely connected to positive relevant $\mathbf{R}^+$, which is undecidable.
    - Und. of $\mathbf{R}^+$ was shown by Urquhart (1984), but $\mathbf{S}$ eluded these techniques.
    - Eventually, this led Urquhart (2016) to conjecture that $\mathbf{S}$ is decidable.

What we notice about the problem:

- *Valuations are arbitrary*, contra positive intuitionistic logic. ['suggesting' undecidability]
- $\mathbf{S}$ is positive, *no negation*! [suggesting decidability]
- Frames of $\mathbf{S}$ are semilattices, *they are associative*! [suggesting undecidability]

Theorem: $\mathbb{S}$ is undecidable

Relevant S is undecidable: Proof idea

Theorem: S is undecidable.

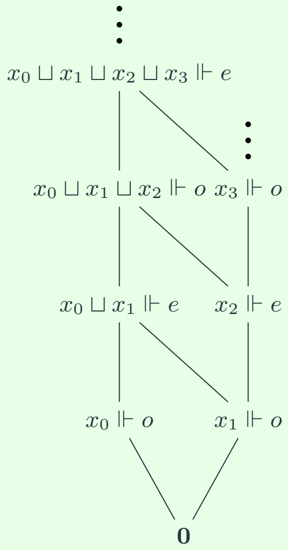
We cover the no-FMP proof instead, since it is considerably simpler than the undecidability proof, yet effectively illustrates some of the same key ideas.[3]

Theorem: S lacks the FMP.

Proof. We show that the formula ψ_∞ from the handout only is refuted by infinite models.

[3] Additionally, it addresses an open problem (as recently raised in Weiss 2021)

Refuting model



Conclusion

We obtained new (undecidability) results, including:

- Hyperboolean modal logic is undecidable.[4]
- Modal logic of semilattices is undecidable.[5]
- $\mathbf{S}$ is undecidable.[6]

We compared them with known decidability results:

- Propositional team logics are decidable.
- Modal information logic is decidable [cf. SBK 2023b].
- Truthmaker logics are decidable [cf. SBK 2023a].

Core messages:

- Valuations matter.
- Associativity matters.
- Negation matters, but we only needed a tiny bit of meta-language 'it is not the case that'.

[4] Raised in Goranko and Vakarelov 1999
[5] Raised in Bergman 2018; Jipsen et al. 2021; SBK 2023b
[6] Raised in Urquhart 1972, 1984

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Thank you!